# On product-one sequences with congruence conditions over non-abelian groups

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**Abstract**

Let $G$ be a finite group. For a positive integer $d$, let $\mathsf{s}_{d\mathbb{N}}(G)$ denote the smallest integer $\ell$ such that every sequence $S$ over $G$ of length $|S| \geq \ell$ has a nonempty product-one subsequence $T$ with $|T| \equiv 0 \pmod d$. In this paper, we mainly study this invariant for dihedral groups $D_{2n}$ and metacyclic groups $C_p \ltimes_s C_q$.

*Keywords:* product-one sequence, dihedral groups, metacyclic groups, congruence conditions.

## 1. Introduction

Let $G$ be a finite multiplicative group and let $\exp(G) = \operatorname{lcm}\{\operatorname{ord}(g) \colon g \in G\}$ be the exponent of $G$. By a sequence $S$ over $G$, we mean a finite unordered sequence with terms from $G$ and repetition allowed. We say $S$ is a product-one sequence if its terms can be ordered so that their product equals the identity element of $G$. In most of the cases, a direct "zero-sum" problem asks for the the smallest integer $\ell \in \mathbb{N}$ such that every sequence $S$ over $G$ with length $|S| \geq \ell$ has a product-one subsequence with prescribed length.

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Let $L \subset \mathbb{N}$ be nonempty subset and let $\mathsf{s}_L(G)$ be the smallest $\ell \in \mathbb{N} \cup \{\infty\}$ such that every sequence $S$ over $G$ has a product-one subsequence $T$ with length $|T| \in L$. Thus the classic zero-sum invariants $\mathsf{D}(G) = \mathsf{s}_{\mathbb{N}}(G)$ (the Davenport constant), $\mathsf{s}(G) = \mathsf{s}_{\{\exp(G)\}}(G)$ (the EGZ constant), and $\eta(G) = \mathsf{s}_{[1,\exp(G)]}(G)$. The readers may want to consult one of the surveys or monographs ([9, 16, 13, 17]). Moreover, $\mathsf{s}_L(G)$ is also investigated for various other sets (see, e.g. [8, 19, 3, 10, 11]). Among others, A. Geroldinger et al. [14] introduced $\mathsf{s}_{d\mathbb{N}}(G)$ for finite abelian groups and obtained the following result.

**Theorem A.** *Let $G$ be a finite abelian group and let $d$ be a positive integer.*

1. *Suppose $G$ is cyclic. Then*

$$\mathsf{s}_{d\mathbb{N}}(G) = \mathrm{lcm}(|G|, d) + \gcd(|G|, d) - 1.$$

2. *Suppose $G \cong C_m \oplus C_n$, where $m, n \in \mathbb{N}$ with $1 < m|n$. Then*

$$\mathsf{s}_{d\mathbb{N}}(G) = \mathrm{lcm}(n, d) + \gcd(n, \mathrm{lcm}(m, d)) + \gcd(m, d) - 2.$$

In the present paper, we mainly focus on $\mathsf{s}_{d\mathbb{N}}(G)$ for non-abelian groups. The study of sequences for non-abelian groups dates back to the 1970s (see [23]), and fresh impetus came from applications in factorization theory and invariant theory (see [18, 15, 5, 4, 7]). Diheral groups, dicyclic groups, and metacyclic groups are the most studied ones. Our main results are the following.

**Theorem 1.** *Let $D_{2n}$ be a dihedral group,where $n \geq 3$, and let $d$ be positive integer. Then*

$$\mathsf{s}_{d\mathbb{N}}(D_{2n}) = \begin{cases} 2d + \lfloor \log_2 n \rfloor, & \textit{if } n \mid d \textit{ and } d \textit{ is odd}, \\ d + n = \mathsf{s}_{\{d\}}(D_{2n}), & \textit{if } n \mid d \textit{ and } d \textit{ is even}, \\ nd + 1, & \textit{if } \gcd(n, d) = 1 . \end{cases}$$

*Remark:* Note that $n$ divides $\exp(D_{2n})$ and $\exp(D_{2n})$ is always even. We have

$$\mathsf{s}_{k\exp(D_{2n})\mathbb{N}}(D_{2n}) = \mathsf{s}_{k\exp(D_{2n})}(D_{2n}) = k\exp(D_{2n}) + n\,.$$

**Theorem 2.** *Let $C_p \ltimes_s C_q = \langle x, y : x^p = y^q = 1, yx = xy^s, \mathrm{ord}_q(s) = p,$ and $p, q$ are primes$\rangle$ be a metacyclic group. Then*

$$\mathsf{s}_{kp\mathbb{N}}(C_p \ltimes_s C_q) = \mathrm{lcm}(kp, q) + p - 2 + \gcd(kp, q)\,,$$

*where $k \in \mathbb{N}$.*

## 2. Preliminaries

Our notation and terminology are consistent with [6, 17, 18]. We briefly gather some key notions and fix notation. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we let $[a, b] = \{x \in \mathbb{Z} : a \le x \le b\}$ be the discrete interval between $a$ and $b$. For positive integers $m$ and $n$, we denote by $\gcd(m,\ n)$ and $\mathrm{lcm}(m,\ n)$ the greatest common divisor and the least common multiple of $m,\ n$ respectively. If $\gcd(n, m) = 1$, we let $\mathrm{ord}_n(m)$ be the minimal positive integer $\ell$ such that $g^\ell \equiv 1 \pmod n$.

Let $G$ be a multiplicatively written finite group with identity $1_G \in G$ and let $A, B$ be two nonempty subsets of $G$. We denote $A \cdot B = \{a \cdot b \colon a \in A, b \in B\}$. Let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis $G$. In combinatorial language, elements of $\mathcal{F}(G)$ are called sequences over $G$, which are unordered finite sequences of terms from $G$ with repetition allowed. In order to distinguish between the group operation in $G$ and the sequence operation in $\mathcal{F}(G)$, we use a bold dot symbol $\bullet$ for the multiplication in $\mathcal{F}(G)$, so $G = (G, \cdot)$ and $\mathcal{F}(G) = (\mathcal{F}(G), \bullet)$. In order to avoid confusion between exponentiation of the group operation $\cdot$ in $G$ and exponentiation of the sequence operation $\bullet$ in $\mathcal{F}(G)$, we use brackets to denote exponentiation in $\mathcal{F}(G)$. Thus, for $g \in G$, $T \in \mathcal{F}(G)$, and $k \in \mathbb{N}$, we have $g^{[k]} = \underbrace{g \bullet \ldots \bullet g}_{k}$ and $T^{[k]} = \underbrace{T \bullet \ldots \bullet T}_{k}$. Let

$$S = g_1 \bullet \ldots \bullet g_\ell = \prod_{g \in G}^{\bullet} g^{[\mathsf{v}_g(S)]} \in \mathcal{F}(G)$$

be a sequence over $G$. Then $\mathsf{v}_g(S) \in \mathbb{N}_0$ is the multiplicity of $g$ in $S$,

$|S| = \ell = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ is the length of $S$;

$\mathsf{h}(S) = \max\{\mathsf{v}_g(S) :\ g \in G\}$ is the maximum multiplicity of $S$;

$\mathrm{supp}(S) = \{g \in G :\ \mathsf{v}_g(S) > 0\} \subseteq G$ is the support of $S$;

$\pi(S) = \{g_{\tau(1)} \cdot \ldots \cdot g_{\tau(\ell)} \in G \colon \tau \text{ is a permutation of } [1, \ell]\} \subset G$ is the *set of products* of $S$.

If $|S| = 0$, then we say $S$ is empty and use the convention that $\pi(S) = \{1_G\}$. We denote $S^{-1} = g_1^{-1} \bullet \ldots \bullet g_\ell^{-1}$ and $S_A = \prod^{\bullet}_{g \in A} g^{[\mathsf{v}_g(A)]}$ for a subset $A \subset G$. Note that $\gcd(S_1, S_2) = \prod^{\bullet}_{g \in G} g^{[\min\{\mathsf{v}_g(S_1), \mathsf{v}_g(S_2)\}]} \in \mathcal{F}(G)$ for any

$S_1, S_2 \in \mathcal{F}(G)$. For $n \in \mathbb{N}$, the *n-products* and *sequence subproducts* of $S$ are respectfully denoted by

$$\Pi_n(S) = \bigcup_{T|S, |T|=n} \pi(T) \subset G \quad \text{and} \quad \Pi(S) = \bigcup_{n\geq 1} \Pi_n(S) \subset G\,.$$

In addition, we write

$$\Pi_{\leq k}(S) = \bigcup_{j\in[1,k]} \Pi_j(S) \ \text{ and } \ \Pi_{\geq k}(S) = \bigcup_{j\geq k} \Pi_j(S).$$

We say $S$ is

- squarefree if $\mathsf{v}_g(S) \leq 1$ for all $g \in G$;
- a subsequence of $W$ if $W$ is a sequence over $G$ with $\mathsf{v}_g(W) \geq \mathsf{v}_g(S)$ for all $g \in G$ (Since $S$ divides $W$ in $\mathcal{F}(G)$, we denote it by $S \mid W$);
- a product-one sequence if $1_G \in \pi(S)$;
- product-one free if $1_G \notin \Pi(S)$;
- a minimal product-one sequence if $S$ is a product-one sequence and $S = T_1 \cdot T_2$ implies that $T_1$ or $T_2$ is empty, where $T_1, T_2$ are two product-one sequences.
- a $\pm$-product-one sequence if there exist a permutation $\tau$ of $[1, \ell]$ and $\varepsilon_i \in \{\pm 1\}$ for $1 \leq i \leq \ell$ such that $g_{\tau(1)}^{\varepsilon_1} \cdot g_{\tau(2)}^{\varepsilon_2} \cdot \ldots \cdot g_{\tau(\ell)}^{\varepsilon_\ell} = 1_G$

The following four lemmas collect some well-known results in additive combinatorics, which we will need later.

**Lemma 3 ([21, Lemma 2.2]).** *Let $A$, $B$ be two nonempty subsets of a finite group $G$. If $|A| + |B| > |G|$, then $A \cdot B = G$.*

**Lemma 4.** *Let $C_n$ be a cyclic group of order $n$ and let $D_{2n}$ be a dihedral group of order $2n$, where $n \geq 3$.*

1. $\mathsf{D}(C_n) = \mathsf{s}_{\mathbb{N}}(C_n) = n$ *and every minimal product-one sequence of length $n$ over $C_n$ must have the form $S = g^{[n]}$, where $g \in C_n$ with* $\operatorname{ord}(g) = n$ *([16, Theorem 5.1.10.1]).*

2. $\mathsf{s}_{\{n\}}(C_n) = 2n-1$ *(Erdős-Ginzburg-Ziv Theorem, see e.g. [16, Corollary 5.7.5]).*
3. $\mathsf{D}(D_{2n}) = \mathsf{s}_{\mathbb{N}}(D_{2n}) = n+1$ *([2, Lemma 4]).*
4. $\mathsf{s}_{\{2n\}}(D_{2n}) = 3n$ *([2, Theorem 8]).*
5. *If $n$ is even, then $\mathsf{s}_{\{n\}}(D_{2n}) = 2n$ ([22, Theorem 1.1.1]).*

**Lemma 5.** *Let $G$ be a finite abelian group of order $n$ and let $S$ be a sequence over $G$*

1. *If $|S| \ge n$, then $S$ has a nonempty product-one subsequence of length $|S| \le \mathsf{h}(S)$ ([16, Theorem 5.7.3]).*
2. *If $r = |S|-(n-2) \ge 2$ and $S$ has no product-one subsequence of length $n$, then $|\Pi_{n-2}(S)| = |\Pi_r(S)| \ge r-1$ ([12, Lemma 7]).*
3. *If $S$ is product-one free, then $|\Pi(S)| \ge |S| + |\operatorname{supp}(S)| - 1$ ([16, Proposition 5.3.5.1]).*

**Lemma 6 ([1, Lemma 2.1]).** *Let $G$ be a multiplicative cyclic group of order $n$ and let $S$ be a sequence over $G$ of length $\ge \lfloor \log_2 n \rfloor + 1$. Then $S$ has a nonempty $\pm$-product-one subsequence, which means there exist a subset $J \subset [1, |S|]$ such that*

$$\prod_{j\in J} g_j = \prod_{i\in[1,|S|]\setminus J} g_i\,,$$

*provided that $S = g_1 \boldsymbol{\cdot} g_2 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} g_{|S|}$.*

We also need the following three technical lemmas.

**Lemma 7 ([24, Theoem 1.1]).** *Let $G$ be a multiplicative cyclic group of order $n$ and let $S$ be a sequence over $G$ of length $n-1$. If $\Pi(S) \ne G$ and for every subgroup $H \subsetneq G$, we have $|S_H| \le |H|-1$, then there exists $g \in G$ with $\operatorname{ord}(g) = n$ such that $S = g^{[n-1]}$.*

**Lemma 8.** *Let $k, h \in \mathbb{N}$ and let $a_1, \ldots, a_k \in \mathbb{Z}$ with $|a_i| \le h$ for every $i \in [1,k]$. Then there exists a subset $I \subset [1,k]$ such that $0 \le \sum_{i\in I} a_i - \sum_{j\in[1,k]\setminus I} a_j \le h$.*

Proof. We proceed by induction on $k$. If $k = 1$, then the assertion is trivial. Suppose $k \ge 2$ and the assertion holds for $k-1$. Then there exists $I_1 \subset [1, k-1]$ such that $0 \le \sum_{i\in I_1} a_i - \sum_{j\in[1,k-1]\setminus I_1} a_j \le h$. Let $d = \sum_{i\in I_1} a_i -$

$\sum_{j\in[1,k-1]\setminus I_1} a_j$. If $a_k \ge 0$, then $|\sum_{i\in I_1} a_i - \sum_{j\in[1,k]\setminus I_1} a_j| = |d-a_k| \le h$ and hence the assertion follows by choosing $I = I_1$ or $[1,k]\setminus I_1$. If $a_k < 0$, then $|\sum_{i\in I_1\cup\{k\}} a_i - \sum_{j\in[1,k-1]\setminus I_1} a_j| = |d+a_k| \le h$ and hence the assertion follows by choosing $I = I_1 \cup \{k\}$ or $[1,k-1]\setminus I_1$. □

**Lemma 9.** *Let $G$ be a cyclic group of order $n$ with $n \ge 3$ odd and let $S$ be a sequence over $G$ of length $\ge n$.*

1. *$S$ has a $\pm$-product-one subsequence $T$ of odd length with $|T| \le n$.*
2. *If $S$ has no $\pm$-product-one subsequence $T$ of odd length with $|T| < n$, then there exists $g \in G$ with $\mathrm{ord}(g) = n$ and $r \in [0, |S|]$ such that $S = g^{[r]} \boldsymbol{\cdot} (-g)^{[|S|-r]}$.*

Proof. Since $G$ is abelian, for every subsequence $T$ of $S$, $\pi(T)$ has only one element and we denote such an element by $\sigma(T)$, whence $\pi(T) = \{\sigma(T)\}$.

1. Let $H$ be a minimal subgroup $G$ such that $|S_H| \ge |H|$. If $H$ is trivial, then the assertions follows immediately. Now suppose $|H| \ge 3$ and let $T_H$ be a subsequence of $S_H$ with length $|H|$. It suffices to show $T_H$ has a $\pm$-product-one subsequence $T$ of odd length. Thus we may assume $G = H$ and $|S| = n$. Fix one term $g_0$ of $S$ and set $S_1 = S \boldsymbol{\cdot} g_0^{[-1]}$,

$$\Pi_E(S) = \bigcup_{n \text{ is even}} \Pi_n(S), \qquad \Pi_O(S) = \bigcup_{n \text{ is odd}} \Pi_n(S)\,,$$
$$\Pi_E(S_1) = \bigcup_{n \text{ is even}} \Pi_n(S_1), \qquad \Pi_O(S_1) = \bigcup_{n \text{ is odd}} \Pi_n(S_1)\,.$$

Suppose $\Pi_E(S) \cap \Pi_O(S) \ne \emptyset$. Then there exist subsequences $T_1, T_2$ of $S$ with $|T_1|$ odd and $|T_2|$ even such that $\sigma(T_1) = \sigma(T_2)$, whence

$$\sigma(T_1 \boldsymbol{\cdot} (\gcd(T_1,T_2))^{[-1]}) = \sigma(T_2 \boldsymbol{\cdot} (\gcd(T_1,T_2))^{[-1]})\,.$$

It follows that $T_1 \boldsymbol{\cdot} T_2 \boldsymbol{\cdot} (\gcd(T_1,T_2))^{[-2]}$ is $\pm$-product-one subsequence of odd length.

Suppose $\Pi(S_1) \ne G$. It follows by Lemma 7 that $S = g^{[n-1]}$ for some $g \in G$ with $\mathrm{ord}(g) = n$. Then there exists $x \in [0, n-1]$ such that $g_0 = g^x$. If $x$ is odd, then $g^{[n-x]} \boldsymbol{\cdot} g_0$ is a product-one sequence of odd length. If $x$ is even, then $g^{[x]} \boldsymbol{\cdot} g_0$ is a $\pm$-product-one sequence of odd length.

Assume to the contrary that $\Pi_E(S) \cap \Pi_O(S) = \emptyset$ and $\Pi(S_1) = G$. Then

$$\Pi_E(S_1) \bigcup g_0\Pi_O(S_1) \subset \Pi_E(S) \ \text{ and } \ \Pi_O(S_1) \bigcup g_0\Pi_E(S_1) \subset \Pi_O(S)\,,$$

whence

$$\begin{aligned} n=|G| &\le |\Pi_O(S_1)|+|\Pi_E(S_1)| \le |\Pi_O(S)|+|\Pi_E(S)| \le n \quad \text{and} \\ n=|G| &\le |g_0\Pi_O(S_1)|+|g_0\Pi_E(S_1)| \le |\Pi_E(S)|+|\Pi_O(S)| \le n\,. \end{aligned}$$

Therefore $|\Pi_O(S_1)| = |\Pi_O(S)| = |g_0\Pi_E(S_1)| = |\Pi_E(S_1)|$ and hence $n = 2|\Pi_E(S_1)|$ is even, a contradiction.

2. Let $S_0$ be a subsequence of $S$ with length $n$. We first show that $S_0$ has the asserted form.

Suppose $S_0$ is a product-one sequence. If $S_0$ is not a minimal product-one sequence, then $S_0 = S_1 \boldsymbol{\cdot} S_2$, where $S_1, S_2$ are nonempty product-one subsequences, whence $|S_1|$ or $|S_2|$ must be odd, a contradiction to our assumption. Thus $S$ is a minimal product-one sequence and it follows by Lemma 4.1 that $S_0 = g^{[n]}$ for some $g \in G$ with $\mathrm{ord}(g) = n$.

Suppose $S_0$ is not a product-one sequence. By 1., $S_0$ must be a $\pm$-product-one subsequence and hence $S_0 = T_1 \boldsymbol{\cdot} T_2$ with $\sigma(T_1) = \sigma(T_2)$, where $T_1, T_2$ are nonempty sequences.

If there exist subsequences $T_1' \mid T_1$ and $T_2' \mid T_2$ with $1 \le |T_1' \boldsymbol{\cdot} T_2'| < |S_0|$ such that $\sigma(T_1') = \sigma(T_2')$, then both $T_1' \boldsymbol{\cdot} T_2'$ and $S_0 \boldsymbol{\cdot} (T_1' \boldsymbol{\cdot} T_2')^{[-1]}$ are nonempty $\pm$-product-one subsequences of length $< n$. Thus, by our assumption, both $|T_1' \boldsymbol{\cdot} T_2'|$ and $|S_0 \boldsymbol{\cdot} (T_1' \boldsymbol{\cdot} T_2')^{[-1]}|$ are even, a contradiction to the fact that $|S_0| = n$ is odd. Therefore $T_1, T_2$ are product-one free and $\Pi(T_1) \cap \Pi(T_2) = \{\sigma(T_1)\}$, whence $|\pi(T_1)| \ge |T_1|$ and $|\pi(T_2)| \ge |T_2|$ by Lemma 5.3.

It follows that

$$\begin{aligned} n-1 &\ge |\Pi(T_1) \cup \Pi(T_2)| = |\Pi(T_1)| + |\Pi(T_2)| - |\Pi(T_1) \cap \Pi(T_2)| \\ &\ge |T_1| + |T_2| - 1 = n-1\,, \end{aligned}$$

whence $|\Pi(T_1)| = |T_1|$, $|\Pi(T_2)| = |T_2|$, and $\Pi(T_1) \setminus \{\sigma(T_1)\} = G \setminus \Pi(T_2)$. Thus by Lemma 5.3 again, we have $|\operatorname{supp}(T_1)| = |\operatorname{supp}(T_2)| = 1$, which implies that there exist $g_1, g_2 \in G$ with $\mathrm{ord}(g_1) > |T_1|$ and $\mathrm{ord}(g_2) > |T_2|$ such that $T_1 = g_1^{[|T_1|]}$ and $T_2 = g_2^{[|T_2|]}$. By symmetry, we may assume that $|T_1| < |T_2|$. Thus $|T_2| > n/2$ and hence $\mathrm{ord}(g_2) = n$. Let $r \in [1, n-1]$ such that $g_1 = g_2^r$. Then $\sigma(T_1) = \sigma(T_2)$ implies $|T_2| = n - |T_1| \equiv r|T_1| \pmod n$. Note that

$$\Pi(T_2) = \{g_2^i \colon i \in [1, |T_2|]\} \text{ and } \Pi(T_1) = \{g_2^{ir} \colon i \in [1, |T_1|]\}\,.$$

It follows by $\Pi(T_1) \setminus \{\sigma(T_1)\} = G \setminus \Pi(T_2)$ that

$$\Pi(T_1) = \{g_2^{ir} \colon i \in [1, |T_1|]\} = \{g_2^i \colon i \in [|T_2|, n-1]\}\,,$$

whence $r \in [|T_2|, n-1]$ and $r > n/2$. Assume to the contrary that $r \neq n-1$. Then there exists $t \in [2, |T_1|]$ such that $g_2^{tr} = g_2^{n-1}$ and $g_2^{(t-1)r} = g_2^{n-1-r} \in \Pi(T_1)$, whence $n-r-1 \geq |T_2| > n/2$. Thus $n > r+(n-r-1) > n/2+n/2 = n$, a contradiction. Therefore $r = n-1$ and hence $S_0 = (g_2^{-1})^{[k]} \cdot g_2^{[n-k]}$, where $k = |T_1| \in [1, n-1]$.

Now we showed $S_0 = g^{[k]} \cdot (g^{-1})^{[n-k]}$, where $k \in [0, n]$ and $g \in G$ with $\mathrm{ord}(g) = n$. Since $S_0$ is chosen arbitrary, we obtain $\mathrm{supp}(S) = \mathrm{supp}(S_0)$ and hence $S = g^{[k_1]} \cdot (g^{-1})^{[|S|-k_1]}$, where $k_1 \in [0, |S|]$. □

## 3. The proof of Theorem 1

Throughout the whole section, we consider the dihedral group $D_{2n} := \langle x, y : x^2 = y^n = 1, xy = y^{-1}x \rangle$, and let $H = \langle y \rangle$ and $N = D_{2n} \setminus H$, where $n \geq 3$.

**Lemma 10 ([20], Theorem 1.3).** *Let $S$ be a product-one free sequence of length $n$ over the dihedral group $D_{2n}$, where $n \geq 3$. If $|S_N| \geq 2$, then $n = 3$ and $S = x \cdot xy \cdot xy^2$.*

**Lemma 11.** *Let $n \geq 3$ be a positive integer. Then*

$$\mathsf{s}_{[1,n]}(D_{2n}) = n + 1.$$

Proof. It is easy to see that $W = x \cdot y^{[n-1]}$ is a product-one free sequence of length $n$ over $D_{2n}$. Thus $\mathsf{s}_{[1,n]}(D_{2n}) \geq n+1$. Let $S$ be a sequence of length $n+1$ over $D_{2n}$. It suffices to show $S$ has a product-one subsequence $T$ of length $1 \leq |T| \leq n$.

If $|S_H| \geq n$, then the assertion follows by Lemma 4.1. If $|S_H| \leq n-1$, then $|S_N| = |S| - |S_H| \geq 2$. Assume to the contrary that $S$ has no product-one subsequence $T$ of length $1 \leq |T| \leq n$. Then $1_G \notin \mathrm{supp}(S)$. Let $W$ be a subsequence of $S$ with length $n$ such that $|W_N| \geq 2$. It follows from Lemma 10 that $n = 3$ and $W = x \cdot xy \cdot xy^2$. Set $S \cdot W^{[-1]} = y^{\alpha}$, where $\alpha \in [1, 2]$. Therefore $x \cdot y^{\alpha} \cdot xy^{\alpha}$ is a product-one subsequence of length $n$, a contradiction. □

**Lemma 12.** *If $S$ is a $\pm$-product-one sequence over $D_{2n}$ with $|S_N| \geq 1$, where $n \geq 3$, then $S$ is a product-one sequence.*

Proof. Since $S$ is a $\pm$-product-one sequence, we obtain $|S_N|$ is even. Suppose $S = xy^{\alpha_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} xy^{\alpha_{2u}} \boldsymbol{\cdot} y^{\beta_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} y^{\beta_k}$, where $u \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\alpha_1, \ldots, \alpha_{2u}, \beta_1, \ldots, \beta_k \in \mathbb{Z}$. Note that $(xy^{\alpha_i})^{-1} = xy^{\alpha_i}$ for every $i \in [1, 2u]$. After renumbering if necessary, there exists $v \in [1, k]$ such that

$$S' = xy^{\alpha_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} xy^{\alpha_{2u}} \boldsymbol{\cdot} y^{\beta_1} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} y^{\beta_v} \boldsymbol{\cdot} y^{-\beta_{v+1}} \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} y^{-\beta_k}$$

is a product-one sequence. It follows from $\pi(S) = \pi(S')$ that $S$ is also a product-one sequence. $\square$

The following proposition is crucial in the proof of Theorem 1.

**Proposition 13.** *Let $n$ be an odd integer with $n \geq 3$. Then*

$$\mathsf{s}_{n\mathbb{N}}(D_{2n}) = 2n + \lfloor \log_2 n \rfloor .$$

Proof. Let $W = x^{[2n-1]} \boldsymbol{\cdot} \prod^\bullet{}_{i=0}^{\lfloor \log_2 n \rfloor - 1} y^{2^i}$ be a sequence of length $2n + \lfloor \log_2 n \rfloor - 1$ over $D_{2n}$. Since $n$ is odd, we obtain $W$ has no nonempty product-one subsequence $T$ of length $|T| \equiv 0 \pmod n$. Thus $\mathsf{s}_{n\mathbb{N}}(D_{2n}) \geq 2n + \lfloor \log_2 n \rfloor$.

Let $S$ be a sequence of length $2n + \lfloor \log_2 n \rfloor$. It suffices to show $S$ has a product-one subsequence of length $n$ or $2n$. If $S_H$ has a product-one subsequence of length $n$, then we are done. Thus we may assume that $S_H$ has no product-one subsequence of length $n$. It follows from Lemma 5.2 that

$$|\Pi_{n-2}(S_H)| \geq |S_H| - (n-1) \tag{1}$$

and by Lemma 4.2 that $|S_H| \leq 2n-2$, which implies $|S_N| \geq \lfloor \log_2 n \rfloor + 2 \geq 3$.

By changing the generators if necessary, we may assume that $\mathsf{v}_x(S_N) = \mathsf{h}(S_N)$. If $\mathsf{h}(S_N) = 1$, then $S_N$ is squarefree and hence $|\Pi_2(S_N)| \geq |S_N| - 1$. In view of Equation (1), we have $|\Pi_2(S_N)| + |\Pi_{n-2}(S_H)| \geq |S_N| - 1 + |S_H| - (n-1) = n + \lfloor \log_2 n \rfloor > |H|$. Note that $\Pi_2(S_N) \subset H$. It follows from Lemma 3 that

$$\Pi_n(S) \supset \Pi_2(S_N) \ \cdot \ \Pi_{n-2}(S_H) = H \,,$$

which implies that $S$ has a product-one subsequence of length $n$.

Now suppose $\mathsf{h}(S_N) = \mathsf{v}_x(S_N) \geq 2$. Let $\phi \colon D_{2n} \to D_{2n}$ be a map defined by $\phi(xy^\alpha) = y^\alpha$ and $\phi(y^\alpha) = y^\alpha$ for all $\alpha \in [0, n-1]$. Since $\phi(D_{2n}) = H$ is abelian, for every sequence $T$ over $H$, $\pi(T)$ has only one element and we denote such an element by $\sigma(T)$, whence $\pi(T) = \{\sigma(T)\}$. We proceed by the following claim.

**Claim A.** *If either* $\gcd(\phi(S_N), S_H)$ *is nonempty or* $S$ *has a* $\pm$*-product-one subsequence* $T$ *of odd length such that* $|T| \le n$*, then* $S$ *has a product-one subsequence of length* $n$ *or* $2n$*.*

*Proof of Claim A.* We distinguish two cases depending on our assumption.

*Case 1:* The sequence $\gcd(\phi(S_N), S_H)$ is nonempty.

Then $T_0 = x \boldsymbol{\cdot} y^\alpha \boldsymbol{\cdot} xy^\alpha$ is a product-one subsequence of $S$, where $y^\alpha$ is a term of $\gcd(\phi(S_N), S_H)$. If $n = 3$, then we are done. Now suppose $n \ge 5$. Set

$$(S \boldsymbol{\cdot} T_0^{[-1]})_N = U_1^{[2]} \boldsymbol{\cdot} W_1 \quad \text{and} \quad (S \boldsymbol{\cdot} T_0^{[-1]})_H = U_2^{[2]} \boldsymbol{\cdot} E \boldsymbol{\cdot} E^{-1} \boldsymbol{\cdot} W_2,$$

where $U_1, U_2, E, W_1, W_2$ are subsequences such that $W_1, W_2$ are squarefree and $W_2$ has no subsequence of length 2. It follows that $|W_1| \le n$ and $|W_2| \le \frac{n+1}{2}$, which implies that

$$2|U_1|+2|U_2|+2|E| = |S|-|T_0|-|W_1|-|W_2| \ge (n-1)/2+\lfloor \log_2 n \rfloor-3 > 0\,. \quad (2)$$

Let $X$ be a maximal subsequence of $\gcd(\phi(W_1), W_2)$ with even length. Then $|\gcd(\phi(W_1) \boldsymbol{\cdot} X^{[-1]}, W_2 \boldsymbol{\cdot} X^{[-1]})| \le 1$ and $X \boldsymbol{\cdot} \phi^{-1}(X)$ is a product of product-one subsequences of length 4. It follows that $|W_1| + |W_2| - 2|X| \le n + 1$, which implies $|T_0| + 2|U_1| + 2|U_2| + 2|E| + 2|X| \ge 2n + \lfloor \log_2 n \rfloor - (n+1) \ge n$. In view of both $|T_0|$ and $n$ are odd, it follows from Equation (2) that there exist subsequences $U_1' \mid U_1$, $U_2' \mid U_2$, $E' \mid E$, and $X' \mid X$ such that

$$Y := T \boldsymbol{\cdot} (U_1')^{[2]} \boldsymbol{\cdot} (U_2')^{[2]} \boldsymbol{\cdot} E' \boldsymbol{\cdot} (E')^{-1} \boldsymbol{\cdot} X' \boldsymbol{\cdot} \phi^{-1}(X')$$

is a $\pm$-product-one subsequence of length $n$. Since $|(T_0)_N| \ge 1$, Lemma 12 implies that $Y$ is a product-one subsequence of length $n$.

*Case 2:* The sequence $\gcd(\phi(S_N), S_H)$ is empty and there is a $\pm$-product-one subsequence $T$ of $S$ with odd length such that $|T| \le n$.

Among all the choices of $T$, we may assume that $T$ is such a sequence with minimal length.

Suppose $|T| < n$. Let $T_0 = T$ if $|T_N| \ge 1$ and let $T_0 = T \boldsymbol{\cdot} x^{[2]}$ if $|T_N| = 0$. Set

$$(S \boldsymbol{\cdot} T_0^{[-1]})_N = U_1^{[2]} \boldsymbol{\cdot} W_1 \quad \text{and} \quad (S \boldsymbol{\cdot} T_0^{[-1]})_H = U_2^{[2]} \boldsymbol{\cdot} W_2,$$

where $U_1, U_2, W_1, W_2$ are sequences such that $W_1, W_2$ are squarefree. Since $\gcd(\phi(S_N), S_H)$ is empty, we obtain $|W_1| + |W_2| = |\phi(W_1)| + |W_2| \le n$, which implies that $|T_0|+2|U_1|+2|U_2| \ge |S|-n \ge n$. In view of both $|T_0|$ and $n$ are odd, there exist subsequences $U_1'$ of $U_1$ and $U_2'$ of $U_2$ such that $T_0 \boldsymbol{\cdot} (U_1')^{[2]} \boldsymbol{\cdot} (U_2')^{[2]}$ is a $\pm$-product-one subsequence of length $n$. Since $|(T_0)_N| \ge 1$, it follows by Lemma 12 that $T_0 \boldsymbol{\cdot} (U_1')^{[2]} \boldsymbol{\cdot} (U_2')^{[2]}$ is a product-one subsequence of length $n$.

Suppose $|T| = n$. If $|T_N| \ge 1$, then Lemma 12 implies that $T$ is a product-one subsequence of length $n$. If $|T_N| = 0$, then $|S_H| \ge |T| = n$. By the minimality of $|T|$, we obtain $S_H$ has no $\pm$-product-one subsequence $T'$ of odd length with $|T'| < n$. Thus Lemma 9.2 implies that $S_H = g^{[k]} \boldsymbol{\cdot} (g^{-1})^{[|S_H|-k]}$, where $k \in [0, |S|]$ and $g = y^{\alpha}$ with $\gcd(\alpha, n) = 1$.

If $|\operatorname{supp}(S_N)| = 1$, say $S_N = (xy^{\beta})^{[|S_N|]}$, where $\beta \in [0, n-1]$, then there exist $k_1 \in [1, \lfloor |S_N|/2 \rfloor]$, $k_2 \in [0, \lfloor k/2 \rfloor]$, and $k_3 \in [0, \lfloor (|S_H|-k)/2 \rfloor]$ such that $(xy^{\beta})^{[2k_1]} \boldsymbol{\cdot} g^{[2k_2]} \boldsymbol{\cdot} (g^{-1})^{[2k_3]}$ is a product-one subsequence of length $2n$.

Suppose $|\operatorname{supp}(S_N)| \ge 2$, say $xg^{\beta_1}, xg^{\beta_2} \in \operatorname{supp}(S_N)$, where $\beta_1, \beta_2 \in [0, n-1]$ with $\beta_1 < \beta_2$. If $\beta_2 - \beta_1$ is even, then let $k_1 \in [0, k]$ and $k_2 \in [0, |S_H| - k]$ such that $k_1 + k_2 = n - (\beta_2 - \beta_1) \le n-2$, whence $W := xg^{\beta_1} \boldsymbol{\cdot} (g^{-1})^{[k_2]} \boldsymbol{\cdot} xg^{\beta_2} \boldsymbol{\cdot} g^{[k_1]}$ is a product-one subsequence of odd length $2 + n - (\beta_2 - \beta_1) \le n$. The minimality of $|T|$ implies that $W$ is a product-one subsequence of length $n$. If $\beta_2 - \beta_1$ is odd, then let $k_1 \in [0, k]$ and $k_2 \in [0, |S_H| - k]$ such that $k_1 + k_2 = \beta_2 - \beta_1$, whence $W := xg^{\beta_1} \boldsymbol{\cdot} g^{[k_2]} \boldsymbol{\cdot} xg^{\beta_2} \boldsymbol{\cdot} (g^{-1})^{[k_1]}$ is a product-one subsequence of odd length $2 + (\beta_2 - \beta_1) \le n$. The minimality of $|T|$ implies that $W$ is a product-one subsequence of length $n$.

$\square$[End of Proof of Claim A.]

By Claim A and Lemma 9.1, we may assume that $|S_H| \le n-1$ and $\gcd(\phi(S_N), S_H)$ is empty. Since $|S_N| \ge n + \lfloor \log_2 n \rfloor + 1$, it follow by using Lemma 5.1 on $\phi(S_N \boldsymbol{\cdot} x^{[-\mathsf{h}(S_N)]})$ repeatedly, we can find subsequences $P_1, \ldots, P_\ell$ of $S_N \boldsymbol{\cdot} x^{[-\mathsf{h}(S_N)]}$ with $P_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} P_\ell$ dividing $S_N \boldsymbol{\cdot} x^{[-\mathsf{h}(S_N)]}$ such that $\phi(P_i)$ are all product-one subsequences of length $|P_i| \le \mathsf{h}(S_N)$ and

$$\mathsf{h}(S_N) + |P_1| + \ldots + |P_\ell| \ge \lfloor \log_2 n \rfloor + 1 \,.$$

Without loss of generality, we may assume that $\ell \in \mathbb{N}_0$ is the minimal integer such that

$$\mathsf{h}(S_N) + |P_1| + \ldots + |P_\ell| \ge \lfloor \log_2 n \rfloor + 1 \,,$$

whence

$$\lfloor \log_2 n \rfloor + 1 \le \mathsf{h}(S_N) + |P_1| + \ldots + |P_\ell| \le \lfloor \log_2 n \rfloor + 1 + \mathsf{h}(S_N) \,.$$

Suppose

$$S_N = x^{[\mathsf{h}(S_N)]} \boldsymbol{\cdot} P_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} P_\ell \boldsymbol{\cdot} U_1^{[2]} \boldsymbol{\cdot} W_1 \quad \text{and} \quad S_H = U_2^{[2]} \boldsymbol{\cdot} W_2,$$

where $U_1, U_2, W_1, W_2$ are subsequences such that $W_1$, $W_2$ are squarefree. It follows from the fact that $\gcd(\phi(S_N), S_H)$ is empty that $|W_1 \boldsymbol{\cdot} W_2| \le n$. By using Lemma 6, we can find subsequences $L_1, \ldots, L_k$ of $W_1 \boldsymbol{\cdot} W_2$ such that $\phi(L_i)$ are $\pm$-product-one sequences with $|L_i| \le \lfloor \log_2 n \rfloor + 1$ and $\phi(W_1 \boldsymbol{\cdot} W_2 \boldsymbol{\cdot} (L_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} L_k)^{[-1]})$ has no $\pm$-product-one sequence, which implies

$$|W_1 \boldsymbol{\cdot} W_2 \boldsymbol{\cdot} (L_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} L_k)^{[-1]}| \le \lfloor \log_2 n \rfloor .$$

Therefore

$$|x^{[\mathsf{h}(S_N)]} \boldsymbol{\cdot} P_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} P_\ell \boldsymbol{\cdot} U_1^{[2]} \boldsymbol{\cdot} U_2^{[2]} \boldsymbol{\cdot} L_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} L_k| \ge 2n . \tag{3}$$

Set $L_i = L_i^{(1)} \boldsymbol{\cdot} L_i^{(2)}$ such that $\sigma(\phi(L_i^{(1)})) = \sigma(\phi(L_i^{(2)}))$ for every $i \in [1, k]$. Now we distinguish two cases.

Suppose there exists $i \in [1, k]$ such that $|(L_i)_H|$ is odd. By symmetry we may assume $|(L_i^{(1)})_N| \ge |(L_i^{(2)})_N|$. Since $|(L_i^{(1)})_N| - |(L_i^{(2)})_N| \le \lfloor \log_2 n \rfloor + 1$, we have

$$|(L_i^{(1)})_N| - |(L_i^{(2)})_N| \le \mathsf{h}(S_N) + |P_1| + \ldots + |P_\ell| .$$

Let $J \subset [1, \ell]$ be a minimal subset (note that $J$ could be empty) such that

$$|(L_i^{(1)})_N| - |(L_i^{(2)})_N| \le \mathsf{h}(S_N) + \sum_{j \in J} |P_j| .$$

It follows by the minimality of $J$ that

$$0 \le d := |(L_i^{(1)})_N| - |(L_i^{(2)})_N| - \sum_{j \in J} |P_j| \le \mathsf{h}(S_N) .$$

Let $(L_i^{(1)})_N = h_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} h_{|(L_i^{(1)})_N|}$ and $(L_i^{(2)})_N \boldsymbol{\cdot} \prod^\bullet_{j \in J} P_j \ \boldsymbol{\cdot} x^{[d]} = f_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} f_{|(L_i^{(1)})_N|}$, where $h_1, \ldots, h_{|(L_i^{(1)})_N|}, f_1, \ldots, f_{|(L_i^{(1)})_N|} \in N$. Then

$$\begin{aligned}
& h_1 \cdot f_1 \cdot \ldots \cdot h_{|(L_i^{(1)})_N|} \cdot f_{|(L_i^{(1)})_N|} \cdot \sigma(\phi((L_i^{(1)})_H)) \cdot \sigma(\phi((L_i^{(2)})_H))^{-1} \\
=& \sigma(\phi((L_i^{(1)})_N)) \cdot \sigma(\phi((L_i^{(2)})_N \boldsymbol{\cdot} \prod\nolimits^\bullet_{j \in J} P_j \ \boldsymbol{\cdot} x^{[d]}))^{-1} \cdot \sigma(\phi((L_i^{(1)})_H)) \cdot \sigma(\phi((L_i^{(2)})_H))^{-1} \\
=& \sigma(\phi((L_i^{(1)})_N)) \cdot \sigma(\phi((L_i^{(2)})_N))^{-1} \cdot \sigma(\phi((L_i^{(1)})_H)) \cdot \sigma(\phi((L_i^{(2)})_H))^{-1} \\
=& \sigma(\phi(L_i^{(1)})) \cdot \sigma(\phi(L_i^{(2)}))^{-1} \\
=& 1_G ,
\end{aligned}$$

which implies that $L_i \cdot x^{[d]} \cdot \prod^\bullet_{j \in J} P_j$ is a $\pm$-product-one subsequence of odd length $|(L_i)_H| + 2|(L_i^{(1)})_N| \le 2|L_i| \le 2\lfloor \log_2 n \rfloor + 2$. Thus $|L_i \cdot x^{[d]} \cdot \prod^\bullet_{j \in J} P_j| \le 2\lfloor \log_2 n \rfloor + 1 \le n$ and hence Claim A implies that $S$ has product-one subsequence of length $n$ or $2n$.

Suppose for all $i \in [1,k]$, we have $|(L_i)_H|$ is even. Let $a_i = |(L_i^{(1)})_N| - |(L_i^{(2)})_N|$ for all $i \in [1,k]$. Then $|a_i| \le \lfloor \log_2 n \rfloor + 1$ and by Lemma 8 there exists a subset $I \subset [1,k]$ such that $0 \le \sum_{i \in I} a_i - \sum_{j \in [1,k] \setminus I} a_j \le \lfloor \log_2 n \rfloor + 1$. Let $L_i' = L_i^{(1)}$ and $L_i'' = L_i^{(2)}$ if $i \in I$; and let $L_i' = L_i^{(2)}$ and $L_i'' = L_i^{(1)}$ if $i \in [1,k] \setminus I$. Set $L' = \prod^\bullet_{i \in [1,k]} L_i'$ and $L'' = \prod^\bullet_{i \in [1,k]} L_i''$. Then

$$\begin{aligned} 0 \le & \sum_{i \in I} a_i - \sum_{j \in [1,k] \setminus I} a_j = |(L')_N| - |(L'')_N| \\ \le & \lfloor \log_2 n \rfloor + 1 \le \mathsf{h}(S_N) + |P_1| + \ldots + |P_\ell| \,. \end{aligned}$$

Let $J_1 \subset [1,\ell]$ be a minimal subset (note that $J_1$ could be empty) such that

$$|(L')_N| - |(L'')_N| \le \mathsf{h}(S_N) + \sum_{j \in J_1} |P_j| \,.$$

It follows by the minimality of $J_1$ that

$$0 \le d_1 := |(L')_N| - |(L'')_N| - \sum_{j \in J_1} |P_j| \le \mathsf{h}(S_N) \,.$$

Again by Lemma 8, there exists a subset $J_2 \subset [1,\ell] \setminus J_1$ such that

$$\begin{aligned} & \left| d_1 + \sum_{j \in J_2} |P_j| - \sum_{j \in [1,\ell] \setminus (J_1 \cup J_2)} |P_j| \right| \\ = & \left| |(L')_N| + \sum_{j \in J_2} |P_j| - |(L'')_N| - \sum_{j \in [1,\ell] \setminus J_2} |P_j| \right| \le \mathsf{h}(S_N) \,. \end{aligned}$$

By symmetry of $(L', J_2)$ and $(L'', [1,\ell] \setminus J_2)$, we may assume that

$$d := |(L')_N| + \sum_{j \in J_2} |P_j| - |(L'')_N| - \sum_{j \in [1,\ell] \setminus J_2} |P_j| \ge 0 \,.$$

Therefore

$$d + |P_1| + \ldots + |P_\ell| \tag{4}$$
$$\leq \begin{cases} \lfloor \log_2 n \rfloor + 1 + \mathsf{h}(S_N) \leq 2\lfloor \log_2 n \rfloor + 1 \leq n, & \text{if } \mathsf{h}(S_N) < \lfloor \log_2 n \rfloor + 1; \\ d = |(L')_N| - |(L'')_N| \leq \lfloor \log_2 n \rfloor + 1 \leq n, & \text{if } \mathsf{h}(S_N) \geq \lfloor \log_2 n \rfloor + 1 . \end{cases}$$

Let $P' = \prod^\bullet_{j \in J_2} P_j$ and $P'' = \prod^\bullet_{j \in [1,\ell] \setminus J_2} P_j$. Suppose $(L')_N \cdot P' = h_1 \cdot \ldots \cdot h_{\ell_0}$ and $(L'')_N \cdot P'' \cdot x^{[d]} = f_1 \cdot \ldots \cdot f_{\ell_0}$, where $\ell_0 = |(L')_N \cdot P'|$ and $h_1, \ldots, h_{\ell_0}, f_1, \ldots, f_{\ell_0} \in N$. Therefore

$$\begin{aligned} & h_1 \cdot f_1 \cdot \ldots \cdot h_{\ell_0} \cdot f_{\ell_0} \cdot \sigma(\phi((L')_H)) \cdot \sigma(\phi((L'')_H))^{-1} \\ =& \sigma(\phi((L')_N \cdot P') \cdot \sigma((L'')_N \cdot P'')^{-1} \cdot \sigma(\phi((L')_H)) \cdot \sigma(\phi((L'')_H))^{-1} \\ =& \sigma(\phi((L')_N)) \cdot \sigma(\phi((L'')_N))^{-1} \cdot \sigma(\phi((L')_H)) \cdot \sigma(\phi((L'')_H))^{-1} \\ =& \sigma(\phi(L')) \cdot \sigma(\phi(L''))^{-1} \\ =& 1_G , \end{aligned}$$

which implies that $W := L' \cdot L'' \cdot x^{[d]} \cdot P' \cdot P''$ is a $\pm$-product-one subsequence of even length. Note that $|W_1 \cdot W_2| \leq n$. In view of Equations (4) and (3), we have

$$|W| \leq |W_1 \cdot W_2| + d + \sum_{i=1}^{\ell} |P_i| \leq 2n \text{ and } |W \cdot x^{[\mathsf{h}(S_N)-d]} \cdot U_1^{[2]} \cdot U_2^{[2]}| \geq 2n .$$

Therefore there exist $k_1 \in [0, \lfloor \mathsf{h}(S_N) - d/2 \rfloor]$, $U_1' \mid U_1$, and $U_2' \mid U_2$ such that $W \cdot x^{[2k_1]} \cdot U_1'^{[2]} \cdot U_2'^{[2]}$ is a $\pm$-product-one sequence of length $2n$, which is also a product-one sequence by Lemma 12 and the fact that $|S_H| \leq n - 1$. □

*Proof of Theorem 1.* We distinguish three cases.

Suppose $d$ is odd and $n|d$. Set $d = kn$, where $k \in \mathbb{N}$. Thus $n$ and $k$ are both odd. Let $W = x^{[2d-1]} \cdot \prod^\bullet{}_{i \in [0, \lfloor \log_2 n \rfloor - 1]} y^{2^i}$ be a sequence of length $2d + \lfloor \log_2 n \rfloor - 1$ over $D_{2n}$. It is easy to see that $W$ has no nonempty product-one subsequence $T$ of length $|T| \equiv 0 \pmod d$. Hence, $\mathsf{s}_{d\mathbb{N}}(D_{2n}) \geq 2d + \lfloor \log_2 n \rfloor$. Let $S$ be a sequence of length $2d + \lfloor \log_2 n \rfloor$ over $D_{2n}$. It suffices to show that $S$ has a product-one subsequence of length $d$ or $2d$. By using Lemma 4.4 on $S$ repeatedly, we have a decomposition

$$S = T_1 \cdot \ldots \cdot T_{k-1} \cdot S_1,$$

where each $T_i$ is a product-one subsequence of length $2n$ and $S_1$ is a sequence of length $2n+\lfloor \log_2 n \rfloor$. It follows from Proposition 13 that $S_1$ has a product-one subsequence $S_2$ of length $n$ or $2n$. If $|S_2|=2n$, then $T_1 \cdot \ldots \cdot T_{k-1} \cdot S_2$ is a product-one subsequence of length $2d$. If $|S_2|=n$, then $T_1 \cdot \ldots \cdot T_{\frac{k-1}{2}} \cdot S_2$ is a product-one subsequence of length $d$.

Suppose $d$ is even and $n|d$. Set $d=kn$, where $k \in \mathbb{N}$. Let $W = 1_G^{[d-1]} \cdot x \cdot y^{[n-1]}$ be a sequence of length $d+n-1$ over $D_{2n}$. It is easy to see that $W$ has no nonempty product-one subsequence $T$ of length $|T| \equiv 0 \pmod d$. Combining the definitions of $\mathsf{s}_{d\mathbb{N}}(D_{2n})$ and $\mathsf{s}_{\{d\}}(D_{2n})$ yields that $\mathsf{s}_{\{d\}}(D_{2n}) \geq \mathsf{s}_{d\mathbb{N}}(D_{2n}) \geq d+n$. Let $S$ be a sequence of length $d+n$ over $D_{2n}$. It suffices to show $S$ has a product-one subsequence of length $d$. If $k$ is even, then by using Lemma 4.4 on $S$ repeatedly we have a decomposition

$$S = T_1 \cdot T_2 \cdot \ldots \cdot T_{\frac{k}{2}} \cdot T',$$

where each $T_i$ is a product-one subsequence of length $2n$ and $T'$ is a sequence of length $n$. Therefore $S \cdot (T')^{[-1]}$ is a product-one subsequence of length $d$. If $k$ is odd, then $n$ is even and by using Lemma 4.5 on $S$ repeatedly we have a decomposition

$$S = T_1 \cdot T_2 \cdot \ldots \cdot T_k \cdot T',$$

where each $T_i$ is a product-one subsequence of length $n$ and $T'$ is a sequence of length $n$. Therefore $S \cdot (T')^{[-1]}$ is a product-one subsequence of length $d$.

Suppose $\gcd(n,d)=1$. Let $W = x \cdot y^{[nd-1]}$ be a sequence of length $nd-1$ over $D_{2n}$. It is easy to see that $W$ has no nonempty product-one subsequence $T$ of length $|T| \equiv 0 \pmod d$. Hence, $\mathsf{s}_{d\mathbb{N}}(D_{2n}) \geq nd+1$. Let $S$ be a sequence of length $nd+1$ over $D_{2n}$. It suffices to show $S$ has a product-one subsequence $T$ of length $|T| \equiv 0 \pmod d$. By using Lemma 11 on $S$ repeatedly, we have a decomposition

$$S = T_1 \cdot \ldots \cdot T_d \cdot T,$$

where each $T_i$ is a product-one subsequence of length $|T_i| \in [1,n]$ and $T$ is a nonempty sequence. Since $|T_1| \cdot \ldots \cdot |T_d|$ is a sequence over $\mathbb{Z}$ of length $d$, it follows by Lemma 4.1 (applied for $\mathbb{Z}/d\mathbb{Z}$) that there exists a subset $I \subset [1,d]$ such that $\sum_{i\in I} |T_i| \equiv 0 \pmod d$. Therefore $S_0 := \prod^\bullet_{i \in I} T_i$ is a product-one subsequence of length $|S_0| \equiv 0 \pmod d$. □

## 4. The proof of Theorem 2

Throughout the whole section, for $p,q$ primes and $s \in [1,q-1]$ with $\mathrm{ord}_q(s)=p$, we consider the metacyclic group $G_{pq} := C_p \ltimes_s C_q = \langle x, y : x^p =$

$y^q = 1_{G_{pq}}, yx = xy^s\rangle$ and let $H = \langle y \rangle$, $N = G_{pq} \setminus H$. We must have $p \ge 2$ and $p \mid q-1$. If $p = 2$, then $G_{pq}$ is a dihedral group of order $2q$. We only consider the case $p \ge 3$, which implies $q \ge 2p+1$. The following lemma will be used in the proof of Theorem 2.

**Lemma 14 ([2], Theorem 15).** *$\mathsf{s}_{\{pq\}}(G_{pq}) = pq + p + q - 2$.*

*Proof of Theorem 2.* Let $W = 1_{G_{pq}}^{[\gcd(kp,q)-1]} \boldsymbol{\cdot} x^{[p-1]} \boldsymbol{\cdot} y^{[\mathrm{lcm}(kp,q)-1]}$ be a sequence of length $\mathrm{lcm}(kp,q) + \gcd(kp,q) + p - 3$ over $G_{pq}$. It is easy to see that $W$ has no nonempty product-one subsequence $T$ of length $|T| \equiv 0 \pmod{kp}$, which implies that $\mathsf{s}_{kp\mathbb{N}}(G_{pq}) \ge \mathrm{lcm}(kp,q) + \gcd(kp,q) + p - 2$. Let $S$ be a sequence of length $\mathrm{lcm}(kp,q) + \gcd(kp,q) + p - 2$ over $G_{pq}$. It suffices to show $S$ has a nonempty subsequence $T$ of length $|T| \equiv 0 \pmod{kp}$.

Set $d = \mathrm{lcm}(kp,q) + \gcd(kp,q) - 1$. If $|S_N| \le p-1$, then $|S_H| \ge d$. It follows from Theorem A that $S_H$ has a nonempty product-one subsequence $T$ of length $|T| \equiv 0 \pmod{kp}$. If $q$ divides $k$, then $|S| = kp + p + q - 2$. By using Lemma 14 on $S$ repeatedly, we have a decomposition

$$S = T_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} T_{\frac{k}{q}} \boldsymbol{\cdot} S_1,$$

where each $T_i$ is a product-one subsequence of length $pq$ and $S_1$ is a sequence of length $p+q-2$. Therefore $S \boldsymbol{\cdot} S_1^{[-1]}$ is a product-one subsequence of length $kp$.

Now we can suppose $|S_N| \ge p$ and $\gcd(q,k) = 1$, which imply $|S| = kpq + p - 1$ and $|S_H| \le kpq - 1$. Let $\psi \colon G_{pq} \to \langle x \rangle$ be the homomorphism defined by $\psi(x^\alpha y^\beta) = x^\alpha$, where $\alpha, \beta \in \mathbb{N}$. Then $\ker \psi = H$. Since $\langle x \rangle \cong C_p$, it follows from Lemma 4.2 that every sequence of length $2p-1$ over $G_{pq}$ has a subsequence $T$ of length $p$ such that $\pi(T) \cap H \ne \emptyset$. Therefore from $S$ we can choose product-one subsequences $A_1, \ldots, A_r$ of length $p$ and subsequences $F_1, \ldots, F_\ell$ of length $p$ with $1_{G_{pq}} \notin \pi(F_i)$ and $\pi(F_i) \cap H \ne \emptyset$ for every $i \in [1,\ell]$ such that

$$A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_r \boldsymbol{\cdot} F_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} F_\ell \mid S \text{ and } |S \boldsymbol{\cdot} (A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_r \boldsymbol{\cdot} F_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} F_\ell)^{[-1]}| \le 2p-2\,,$$

where $r, \ell \in \mathbb{N}_0$. Thus

$$|S \boldsymbol{\cdot} (A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_r \boldsymbol{\cdot} F_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} F_\ell)^{[-1]}| \equiv |S| \equiv p-1 \pmod{p}\,,$$

which implies that $|A_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} A_r \boldsymbol{\cdot} F_1 \boldsymbol{\cdot} \ldots \boldsymbol{\cdot} F_\ell| = kpq$ and $r + \ell = kq$.

If $r \geq k$, then $A_1 \cdot \ldots \cdot A_k$ is a product-one subsequence of length $kp$. Otherwise $r \leq k-1$ and hence $\ell \geq q$. Since $|S_N| \geq p$, there exists $T \in \{A_1, \ldots, A_r, F_1, \ldots, F_\ell\}$ such that $|T_N| \geq 1$. After renumbering if necessary, we may assume that $T \notin \{F_1, \ldots, F_{q-1}\}$. Suppose $T = g_1 \cdot \ldots \cdot g_{p-1} \cdot x^\alpha y^\beta$, where $g_1, \ldots, g_{p-1} \in G_{pq}$ and $x^\alpha y^\beta \in \text{supp}(T_N)$, such that $g_1 \ldots g_{p-1} x^\alpha y^\beta = y^m$ for some $m \in \mathbb{N}_0$, and suppose $y^{m_i} \in \pi(F_i)$ for every $i \in [1, q-1]$, where $m_i \in [1, q-1]$. Thus $y^{m_i} \neq y^{m_i s^\alpha}$ for every $i \in [1, q-1]$ and

$$
\begin{aligned}
& y^m \prod_{i=1}^{q-1} \{y^{m_i}, y^{m_i s^\alpha}\} \\
= & \left\{ y^m \prod_{i \in I} y^{m_i} \prod_{i \in [1,q-1] \setminus I} y^{m_i s^\alpha} : I \subset [1, q-1] \right\} \\
= & \left\{ g_1 \ldots g_{p-1} \left( \prod_{i \in [1,q-1] \setminus I} y^{m_i} \right) x^\alpha y^\beta \prod_{i \in I} y^{m_i} : I \subset [1, q-1] \right\} \\
\subset & \pi(T \cdot y^{m_1} \cdot \ldots \cdot y^{m_{q-1}}) \\
\subset & \pi(T \cdot F_1 \cdot \ldots \cdot F_{q-1}) \,.
\end{aligned}
$$

It follows by the Cauchy-Davenport Theorem (see [21, pp 44-45]) that

$$
|y^m \prod_{i=1}^{q-1} \{y^{m_i}, y^{m_i s^\alpha}\}| \geq \min\{q, 1 + 2(q-1) - (q-1)\} = q \,,
$$

which implies that $H \subset \pi(T \cdot F_1 \cdot \ldots \cdot F_{q-1})$. Thus $1_{G_{pq}} \in \pi(A_1 \cdot \ldots \cdot A_r \cdot F_1 \cdot \ldots \cdot F_\ell)$ and hence $A_1 \cdot \ldots \cdot A_r \cdot F_1 \cdot \ldots \cdot F_\ell$ is a product-one subsequence of length $kpq$. $\square$

## Acknowledgments

This research was supported in part by NSF of China (Grant No. 11671153 and Grant No. 12001331) and by the Austrian Science Fund FWF (Project Number P33499-N).

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